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Reg. No.....

Name.....

M.Sc. DEGREE (C.S.S.) EXAMINATION, MAY 2026

Fourth Semester

Faculty of Science

Branch I (A)–Mathematics

MT 04C 16 – SPECTRAL THEORY

(2016—2018 Admissions–Mercy Chance)

Time : Three Hours

Maximum Weight : 30

Part A

*Answer any **five** questions.
Each question carries 1 weight.*

1. Define a closed linear operator. Give an example.
2. Define strong and weak* convergence of sequence of functionals.
3. Define the terms regular value, resolvent set and spectrum of a linear operator.
4. Define a Banach algebra and give an example.
5. Define the terms E-net and totally bounded set in a metric space X.
6. Define the terms symmetric linear operator and self adjoint linear operator.
7. Show that for any projection P on a Hilbert space H, $\langle px, x \rangle = \|px\|^2$, $p \geq 0$ and $\|p\| \leq 1$.
8. Let $T: \ell^2 \rightarrow \ell^2$ defined by $(x_1, x_2, \dots) \mapsto (0, 0, x_1, x_2, \dots)$. Is T bounded? Is T self adjoint. Justify.

(5 × 1 = 5)

Part B

*Answer any **five** questions.
Each question carries 2 weight.*

9. Prove that in a normal space X, $x_n \xrightarrow{w} x$ if and only if :
 - a) The sequence $(\|x_n\|)$ is bounded.
 - b) For every element f of a total subset $M \subset X'$, we have $f(x_n) \rightarrow f(x)$.





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10. Show that uniform operator convergence $T_n \rightarrow T, T_n \in B(X, Y)$ implies strong operator convergence with the same limit T , where X and Y are Banach spaces.
11. State and prove Banach fixed point theorem.
12. a) Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 1 & 2 \\ -8 & 11 \end{bmatrix}$.
b) Show that the eigen values of a Hermitian matrix are real.
13. Show that the spectrum $\sigma(T)$ of a bounded linear operator $T : X \rightarrow X$ on a complex Banach space X is compact and lies in the disk given by $|\lambda| \leq \|T\|$.
14. Show that if $X \neq \{0\}$ is a complex Banach space and $T \in B(X, X)$, then $\sigma(T) \neq \emptyset$.
15. Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y .
Prove that if (T_n) is uniformly operator convergent, then the limit operator T is compact.
16. Prove that the spectrum $\sigma(T)$ of a bounded self adjoint linear operator $T : H \rightarrow H$ on a complex Hilbert space H is real.

(5 × 2 = 10)

Part C

*Answer any **three** questions.
Each question carries 5 weight.*

17. State and prove open mapping theorem.
18. (a) Prove that if T is a bounded linear operator on a complex Banach space, the spectral radius

$$r_\sigma(T) \text{ of } T \text{ is } r_\sigma(T) = \lim_{n \rightarrow \infty} \sqrt[n]{\|T^n\|}.$$

- (b) Let A be a complex Banach algebra with identity e . Show that for any $x \in A$, the spectrum $\sigma(x) \neq \emptyset$.





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19. (i) Let B be a subset of a metric space X , then prove the following :
- If B is relatively compact, then B is totally bounded.
 - If B is totally bounded and X is complete, then B is relatively compact.
 - If B is totally bounded, then for every $\epsilon > 0$, it has an ϵ -net $M_\epsilon \subset B$.
 - If B is totally bounded, then B is separable.
- (ii) Prove that if X and Y are normed spaces, then the range $R(T)$ of a compact linear operator $T : X \rightarrow Y$ is separable.
20. (a) Define the Hilbert adjoint operator T^* of an operator $T : D(T) \rightarrow H$, where H is a complex Hilbert space.
- (b) Let $S : D(S) \rightarrow H$ and $T : D(T) \rightarrow H$ be linear operators which are densely defined in a complex Hilbert space H . Prove that :
- If $S \subset T$, then $T^* \subset S^*$.
 - If $D(T^*)$ is dense in H , then $T^* \subset T^{**}$.
- (c) Prove that if T is symmetric, then T^{**} is symmetric.
21. Let $T : H \rightarrow H$ be a bounded self adjoint operator on a complex Hilbert space H . Prove that a number λ belongs to the resolved set $\rho(T)$ of T if and only if there exists a $c > 0$ such that for every $x \in H$, $\|T_\lambda(x)\| \geq c\|x\|$.
22. Prove that if two bounded self adjoint linear operators S and T on a Hilbert space H are positive and commute, then their product ST is positive.

(3 × 5 = 15)

