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Reg No :

Name :

M.Sc DEGREE (CSS) EXAMINATION, MARCH 2026

Fourth Semester

M Sc MATHEMATICS

ELECTIVE - ME800401 - DIFFERENTIAL GEOMETRY

2019 Admission Onwards

15579151

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

Answer any **eight** questions.

Weight 1 each.

- Find the level set of the function $f(x_1, x_2) = x_1^2 + x_2^2$ for $c = -2, 0, 2, 5$
- Show that the set of all vectors tangent to $f^{-1}(c)$ at p is contained in the n -dimensional vector subspace $[\nabla f(p)]^\perp$ of $\mathbb{R}_p^{n+1}$ consisting of all vectors orthogonal to $\nabla f(p)$ if $\nabla f(p) \neq 0$.
- Describe the spherical image, when $n = 1$, of $x_1^2 - x_2^2 - \dots - x_{n+1}^2 = 4, x_1 > 0$ oriented by $\mathbf{N} = \frac{\nabla f}{\|\nabla f\|}$.
- Find the velocity, the acceleration and the speed of the parametrized curve $\alpha(t) = (t, t^2)$.
- Define Euclidean and Levi - Civita parallelism.
- Define covariant derivative of the tangent vector field $\mathbf{X}$ with respect to $\mathbf{v} \in S_p$. Show that $D_{\mathbf{v}}(\mathbf{X} + \mathbf{Y}) = D_{\mathbf{v}}\mathbf{X} + D_{\mathbf{v}}\mathbf{Y}$, for all smooth tangent vector fields $\mathbf{X}$ and $\mathbf{Y}$ on S .
- Write a formula for finding curvature of a plane curve at the point p . Also define curvature of a plane curve.
- Let U be an open set in $\mathbb{R}^{n+1}$ and $f : U \rightarrow \mathbb{R}$ is smooth. Then for any parametrized curve $\alpha : I \rightarrow U$ evaluate $\int_{\alpha} df$.
- Define parametrized n -surface in $\mathbb{R}^{n+k}$ ($k \geq 0$). Give an example.





10. Define vector field and coordinate vector fields along a smooth map $\varphi : U \rightarrow \mathbb{R}^{n+k}$, where U open in $\mathbb{R}^n$.

(8×1=8 weightage)

Part B (Short Essay/Problems)

Answer any **six** questions.

Weight 2 each.

11. State and prove Maximal Integral Curve theorem for open subsets in $\mathbb{R}^{n+1}$.
12. For what values of c , the level set $f^{-1}(c)$ is an n -surface, where
- $f(x_1, x_2, \dots, x_{n+1}) = x_1^2 + \dots + x_{n+1}^2$
 - $f(x_1, x_2, \dots, x_{n+1}) = x_1^2 + \dots + x_n^2 - x_{n+1}^2$
13. Verify the following three properties of the differentiation of vector fields along parametrized curves:
- $\mathbf{X} \cdot \dot{\mathbf{Y}} = \dot{\mathbf{X}} \cdot \mathbf{Y}$
 - $f\dot{\mathbf{X}} = f'\mathbf{X} + f\dot{\mathbf{X}}$
 - $(\mathbf{X} \cdot \mathbf{Y})' = \dot{\mathbf{X}} \cdot \mathbf{Y} + \mathbf{X} \cdot \dot{\mathbf{Y}}$
14. Let $\alpha : [0, \pi] \rightarrow S^2$ be the half great circle in S^2 , running from the north pole $p = (0, 0, 1)$ to the south pole $q = (0, 0, -1)$, defined by $\alpha(t) = (\sin t, 0, \cos t)$. Show that, for $\mathbf{v} = (p, v_1, v_2, 0) \in S_p^2$, $P_\alpha(\mathbf{v}) = (q, -v_1, v_2, 0)$
15. Let $\alpha(t) = (x(t), y(t))$, $t \in I$ be a local parametrization of the oriented plane curve C . Show that $\kappa \circ \alpha = \frac{(x'y'' - y'x'')}{(x'^2 + y'^2)^{3/2}}$.
16. Define length of a connected oriented plane curve. Find the length of the connected oriented plane curve $f^{-1}(c)$ oriented by $\frac{\nabla f}{\|\nabla f\|}$, where $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by $f(x_1, x_2) = (x_1 - a)^2 + (x_2 - b)^2$, $c = r^2$
17. Let S be an oriented n -surface in $\mathbb{R}^{n+1}$ and let $p \in S$. Let $\mathbf{Z}$ be any non-zero normal vector field on S such that $\mathbf{N} = \mathbf{Z}/\|\mathbf{Z}\|$ and let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be any basis for S_p . Prove that

$$\mathbf{K}(p) = \frac{(-1)^n \det \begin{pmatrix} \nabla_{\mathbf{v}_1} \mathbf{Z} \\ \vdots \\ \nabla_{\mathbf{v}_n} \mathbf{Z} \\ \mathbf{Z}(p) \end{pmatrix}}{\|\mathbf{Z}(p)\|^n \det \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \\ \mathbf{Z}(p) \end{pmatrix}}$$





18. Let S be a compact connected oriented n -surface in $\mathbb{R}^{n+1}$. Prove that the Gauss – Kronecker curvature $\mathbf{K}(p)$ of S at p is non-zero for all $p \in S$ if and only if the second fundamental form S_p of S at p is definite for all $p \in S$.

(6×2=12 weightage)

Part C (Essay Type Questions)

Answer any **two** questions.

Weight 5 each.

19. Prove the following if S is an n -surface in $\mathbb{R}^{n+1}$ and $\mathbf{X}$ is a smooth tangent vector field on S with $p \in S$. There exists an open interval I containing 0 and a parametrized curve $\alpha : I \rightarrow S$ such that (i) $\alpha(0) = p$ (ii) $\dot{\alpha}(t) = \mathbf{X}(\alpha(t))$ for all $t \in I$ and (iii) If $\beta : \tilde{I} \rightarrow S$ is any other parametrized curve in S satisfying (i) and (ii), then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.
20. Given S is a compact connected oriented n -surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ with $\nabla f(p) \neq 0, \forall p \in S$. Is the Gauss map from S to the unit sphere S^n onto? Explain.
21. Prove that the Weingarten map L_p is self-adjoint; that is, $L_p(\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot L_p(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in S_p$.
22. Prove the following.
- a) Let V be a finite dimensional vector space with dot product and let $L : V \rightarrow V$ be a self adjoint linear transformation on V . Let $S = \{v \in V : v \cdot v = 1\}$ and define $f : S \rightarrow \mathbb{R}$ by $f(v) = L(v) \cdot v$. If f is stationary at $v_0 \in S$, then $L(v_0) = f(v_0) \cdot v_0$.
- b) Let V be a finite dimensional vector space with dot product and let $L : V \rightarrow V$ be a self adjoint linear transformation on V . Then there exists an orthonormal basis for V consisting of eigen vectors of L .

(2×5=10 weightage)

