



QP CODE: 26019085



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Reg No :

Name :

M.Sc DEGREE (CSS) EXAMINATION, MARCH 2026
Fourth Semester
M Sc MATHEMATICS
Core Courses - ME010402 - ANALYTIC NUMBER THEORY

2019 Admission Onwards

ODBCA606

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. State and prove the product formula for $\phi(n)$.
2. Given $f(1) = 1$. Then prove that f is multiplicative if and only if $f(p_1^{a_1} \dots p_r^{a_r}) = f(p_1)^{a_1} \dots f(p_r)^{a_r}$ for all primes p_i and all integers $a_i \geq 1$.
3. If α is a completely multiplicative function then prove that $G(x) = \sum_{n \leq x} \alpha(n) F(\frac{x}{n})$ if and only if $F(x) = \sum_{n \leq x} \mu(n) \alpha(n) G(\frac{x}{n})$.
4. State Shapiro's Taubarian Theorem.
5. Prove that $\forall x \geq 1, \sum_{n \leq x} \vartheta(\frac{x}{n}) = x \log x - x + O(x)$.
6. For a given modulus m prove the following.
 - (a) $\hat{a} = \hat{b}$ if and only if $a \equiv b \pmod{m}$
 - (b) Two integers x and y are in the same residue class if and only if $x \equiv y \pmod{m}$
7. Prove that if a prime p does not divide a then $a^{p-1} \equiv 1 \pmod{p}$.
8. For any prime p , prove that $(p-1)! \equiv -1 \pmod{p}$.
9. Define quadratic residues. Find the quadratic residues for $p=7$.
10. (a) Define $\exp_m(a)$.
 (b) Let $m \geq 1$ and $(a, m) = 1$. Prove that the numbers $1, a, a^2, \dots, a^{f-1}$ are incongruent mod m , where, $f = \exp_m(a)$.

(8×1=8 weightage)





Part B (Short Essay/Problems)

Answer any **six** questions.

Weight **2** each.

11. Prove the following for $x \geq 1$, (a) $\sum_{n > x} \frac{1}{n^s} = O(x^{1-s})$ if $s > 1$.
(b) $\sum_{n \leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha)$ if $\alpha \geq 0$.
12. Prove that $\sum_{n \leq x} \sigma_1(n) = \frac{1}{2} \zeta(2) x^2 + O(x \log x)$ for $x \geq 1$.
Also prove that the average order of $\sigma_1(n)$ is $\frac{\pi^2 n}{12}$.
13. Let $a(n)$ be any arithmetical function and $A(x) = \sum_{n \leq x} a(n)$, where $A(x) = 0$ if $x < 1$. Also f has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Prove that $\sum_{y < n \leq x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t) dt$.
14. Prove that $\frac{1}{6} n \log n < P_n < 12(\log n + n \log \frac{12}{e})$, $\forall n \geq 1$ where P_n is the n^{th} prime.
15. (a) if $a \equiv b \pmod{m}$ then prove that $(a, m) = (b, m)$.
(b) State and prove cancellation law.
16. Assume $m_1, \dots, m_r$ are positive integers, relatively prime in pairs. Let $b_1, \dots, b_r$ be arbitrary integers. Then prove that the system of congruences $x \equiv b_1 \pmod{m_1}, \dots, x \equiv b_r \pmod{m_r}$ has exactly one solution modulo $m_1, \dots, m_r$.
17. Derive a formula for finding $(2|p)$ when p is an odd prime
18. State and prove a necessary and sufficient condition for $\exp_m(a^k) = \exp_m(a)$

(6×2=12 weightage)

Part C (Essay Type Questions)

Answer any **two** questions.

Weight **5** each.

19. (a) Prove that for every $x \geq 1$, $[x]! = \prod_{p \leq x} p^{\alpha(p)}$, where the product is extended over all primes $\leq x$, and $\alpha(p) = \sum_{m=1}^{\infty} \left[\frac{x}{p^m} \right]$.
(b) If $x \geq 2$ prove that $\log[x]! = x \log x - x + O(\log x)$.
(c) Prove that $\sum_{p \leq x} \left[\frac{x}{p} \right] \log p = x \log x + O(x)$ for $x \geq 2$, where the sum is extended over all primes $\leq x$.
20. State and prove any three equivalent forms of the prime number theorem.





21. (a) Let f be a polynomial with integer coefficients, let $m_1, m_2, \dots, m_r$ be positive integer relatively prime in pairs, and let $m = m_1 m_2 \dots m_r$. Prove that the congruence $f(x) \equiv 0 \pmod{m}$ has a solution if and only if each of the congruence $f(x) \equiv 0 \pmod{m_i}$, $i=1, 2, \dots, r$, has a solution.

Also show that if $v(m)$ and $v(m_i)$ denote the number of solutions of $f(x) \equiv 0 \pmod{m}$ and $f(x) \equiv 0 \pmod{m_i}$, $i=1, 2, \dots, r$ respectively, then $v(m) = v(m_1) v(m_2) \dots v(m_r)$.

- (b) Prove that for a given any integer $k > 0$ there exists a lattice point (a, b) such that none of the lattice points $(a+r, b+s)$, $0 < r, s < k$ is visible from the origin.

22. (a) State Gauss' lemma.

- (b) Let m be the number defined in Gauss' lemma. Then prove that

$$m \equiv \sum_{t=1}^{\frac{p-1}{2}} \left[\frac{tn}{p} \right] + (n-1) \frac{p^2-1}{8} \pmod{2}.$$

(2×5=10 weightage)

