QP CODE: 25024350

25024350

Reg No : ....

Name : ....

M.Sc DEGREE (CSS) EXAMINATION, APRIL 2025

Fourth Semester

M Sc MATHEMATICS

CORE - ME010401 - SPECTRAL THEORY

2019 ADMISSION ONWARDS

6F69215F

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Give an example of a sequence of bounded linear operators which is strongly operator convergent but not uniformly operator convergent. Justify your answer.
2. Define closed linear operator. Give an example.
3. Define spectrum of a linear operator $T : D(T) \to X$, where $X \neq \{0\}$ is a complex normed space and $D(T) \subset X$. Also, define the three partitions of the spectrum.
4. If λ is an eigen value of a square matrix A, prove that λ^n is an eigen value of A^n.
5. Find the spectrum of the identity operator and zero operator.
6. Define a commutative algebra. Give an example of an algebra which is not commutative.
7. Prove that the null space of $T_\lambda; \lambda \neq 0$, where T is a compact linear operator on a normed space X is finite dimensional.
8. Let $T : X \to X$ be a compact linear operator on a Banach space X, then prove that every spectral value $\lambda \neq 0$ is an Eigen value of T.
9. Define self-adjoint linear operator on a Hilbert space H. Prove that the eigen values of a bounded self-adjoint linear operator on a complex Hilbert space (if they exists) are real.
10. Define orthogonal projection on a Hilbert space. State and prove any two properties of an orthogonal projection.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Define reflexive spaces. Prove that every Hilbert space H is reflexive.

12. State and prove Uniform Boundedness theorem.

13. Prove that all matrices representing a linear operator $T : X \to X$ on a finite dimensional normed space X relative to various bases for X have the same eigenvalues.

14. If $X \neq \{0\}$ is a complex Banach space and $T \in B(X, X)$, then prove that the spectrum $\sigma(T) \neq \phi$.

15. If $x \in A$ satisfies $\|x\| \leq 1$, then show that $e - x$ is invertible and $(e - x)^{-1} = e + \Sigma_{j=1}^{\infty} x^j$.

16. Prove that a compact linear operator $T : X \longrightarrow Y$, from a normed space X into a Banach space Y has a compact linear extension $\widetilde{T} : \widehat{X} \longrightarrow Y$, where $\widehat{X}$ is the completion of X.

17. Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space $H \neq \{0\}$. Prove that $m, M \in \sigma(T)$ where $m = \inf\limits_{\|x\|=1} \langle Tx, x\rangle$ and $M = \sup\limits_{\|x\|=1} \langle Tx, x\rangle$

18. Let P_1 and P_2 be projections defined on a Hilbert space H and let $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$. Prove that the difference $P = P_2 - P_1$ is a projection if and only if $Y_1 \subset Y_2$.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. (a) Let X and Y be Banach spaces. Let $T : X \to Y$ be a surjective bounded linear operator. Prove that the image $T(B_0)$ of the open unit ball $B_0 = B(0; 1) \subset X$ contains an open ball about $0 \in Y$.
(b) State and prove Bounded Inverse Theorem.

20. State and prove Banach fixed point theorem.

21. (i) Let X and Y be normed spaces and $T : X \to Y$ be a linear operator. Prove that T is compact if and only if T maps every bounded sequence (x_n) in X onto a sequence (Tx_n) in Y which has a convergent subsequence.
(ii) Let $x = (\xi_j) \in l^2$ and $y = (\eta_j) = (\frac{\xi_j}{j})$, for $j = 1, 2, 3, \cdots$. Then show that the operator $T : l^2 \to l^2$ defined by $Tx = y$ is compact.

22. When will be the product of two positive operators on a Hilbert space is positive? Justify your answer.

(2×5=10 weightage)