QP CODE: 25022250

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Reg No : ....

Name : ....

M.Sc DEGREE (CSS) SPECIAL REAPPEARANCE EXAMINATION, APRIL 2025

Third Semester

CORE - ME010304 - FUNCTIONAL ANALYSIS

M.Sc MATHEMATICS , M.Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

86F2770C

Time: 3 Hours   Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Define a complete metric space. Give an example.
2. Define a Cauchy sequence. Give an example.
3. Define a linear operator on a vector space and prove that the range of a linear operator is a vector space.
4. Let T be a bounded linear operator. Then prove that, if $x_n \to x$, where $x_n, x \in D(T)$ implies $Tx_n \to Tx$
5. Define a linear functional in a vector space. Prove that norm is a non-linear functional.
6. State and prove the Pythagorean relation for two orthogonal elements x, y in an inner product space.
7. Write, Euler formulas for finding the fourier coefficients.
8. State Riesz representation theorem.
9. Define Unitary operator. Let U be a unitary operator on a Hilbert space H, prove that U^{-1} is normal.
10. Define partially ordered set. State Zorn's Lemma.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Define translation invariance. Show that the discrete metric on a vector space $X \neq \phi$ cannot be obtained from a norm.

12. Prove that every finite dimensional subspace Y of a normed space X is closed in X.

13. Define a bounded linear operator on a normed space. Give an example for a linear operator which is not bounded. Justify

14. Let X be an n-dimensional vector space and $E = \{e_1, e_2, \dots e_n\}$ a basis for X. Show that $F = \{f_1, f_2, \dots f_n\}$ given by $f_k(c_j) = \delta_{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$ is a basis for the algebraic dual of X.

15. Show that **C[a,b]** is not an inner product space.

16. Let X be an inner product space and $M \neq \phi$ a convex subset which is complete in the metric induced by the inner product. Prove that for any given $x \in X$ there exists a unique $y \in M$ such that $\delta = \inf\{\|x - \tilde{y}\|; \tilde{y} \in M\} = \|x - y\|$.

17. Define the Hilbert-adjoint operator of a bounded linear operator $T : H_1 \to H_2$ where H_1 and H_2 are Hilbert spaces. Prove that the Hilbert-adjoint operator T^* of T is a bounded linear operator with norm $||T^*|| = ||T||$.

18. Define adjoint operator $T^\times$ of bounded linear operator T from a normed space X into a normed space Y. Prove that $T^\times$ is linear, bounded and $||T^\times|| = ||T||$.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. (i) Let X and Y be metric spaces and $T : X \to Y$ a continuous mapping. Prove that the image of a compact subset M of X under T is compact.
(ii) State and prove Riesz's lemma.

20. i)Show that the dual space of R^n is R^n
ii) Show that dual space X^l of a normed space X is a Banach space.

21. Prove that an Orthonormal set M in a Hilbert space H is total in H if and only if for all $x \in H$ the parseval relation holds.

22.
   1. *State and prove Hahn-Banach theorem for normed spaces.*
   2. *For every x in a normed spce X, prove that* $||x|| = \sup_{f \in X', f \neq 0} \frac{|f(x)|}{||f||}$

(2×5=10 weightage)