MSc. Mathematics Degree (MGU-CSS-PG) Examination

MODEL QUESTION PAPER

PC 3 - MT01C03 MEASURE THEORY AND INTEGRATION

Time 3 hrs. Maximum Weight. 30

Section A (Short answer Type Questions) ***Answer any five questions Each question has1 weight***

1) Define the Lebesgue outer measure $m^*(A)$ of a subset A of $\mathbb{R}$ and use the definition to show that the outer measure of a countable set is zero
2) If f is a measurable function and if $f = g$ a.e, show that g is measurable.
3) State the bounded convergence theorem
4) If f is integrable show that $|f|$ is integarble. Is the converse true? Justify.
5) If f is a nonnegative measurable function on a measure space $(X, \mathfrak{B}, \mu)$ show that $\int f \, d\mu = 0$ if and only if $f = 0$ a.e.
6) Define a positive set with respect to a signed measure ν. Show that the union of a countable collection of positive sets is positive.
7) If $f_n \to f$ in measure show that there is a subsequence $\{f_{n_k}\}$ which converge to f a.e.
8) If $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ – finite measure spaces, define the product measure $\mu \times \nu$ on $\mathcal{S} \times \mathcal{T}$

Section B (Short Essay Type Questions) ***Answer any five questions. Each question has 2 weights***

9) Show that the interval of the form (a, ∞) is measurable and use it to show that every Borel set is measurable.
10) If f is measurable and B is a Borel set, show that $f^{-1}(B)$ is measurable
11) State Fatou's' lemma. Give an example to show that we may have strict inequality in the Fatou's lemma.
12) If f and g are integrable over E, show that $f + g$ is integarble over E and
    $\int f + g = \int f + \int g$
13) Let f be an integrable function on a measure space $(X, \mathfrak{B}, \mu)$. Show that given $\varepsilon > 0$ there exists $\delta > 0$ such that for each measurable set E with $m(E) < \delta$, we have $\left|\int_E f\right| < \varepsilon$

14) If ν is a signed measure in a measurable space $(X, \mathfrak{B})$ then show that there is a positive set A and a negative set B such that $A \cap B = \Phi$ and $X = A \cup B$

15) If $f_n \to f$ in measure and if $g_n \to g$ in measure then show that $f_n + g_n \to f + g$ in measure.

16) By integrating $e^{-y} sin2xy$ with respect to x and y, show that $\int_0^\infty \frac{e^{-y} sin2xy}{y} = \frac{1}{4} log5$

Section C (Long Essay Type Questions) ***Answer any three questions. Each question has 5 weights***

17) Show that the outer measure $m^*(A)$ of a subset A of $\mathbb{R}$ satisfies the following properties.

a. $m^*(E) \geq 0$ for every $E \subset \mathbb{R}$

b. $m^*(I) = l(I)$ where I is any interval.

c. $m^*(E) = m^*(E + x)$ for every $E \subset \mathbb{R}$ and $x \epsilon \mathbb{R}$

d. $m^*(\cup E_n) \leq \sum m^*(E_n)$ where $E_n \subset \mathbb{R}$ for every $n = 1,2,3, ...$

e. If $A \subseteq B$ show that $m^*(A) \leq m^*(B)$ for every subset A and B of $\mathbb{R}$

18)

A. Let f be a bounded function defined on a measurable set E with $mE < \infty$. Show that f is measurable if and only if $inf_{f \leq \psi} \int_E \psi(x)dx = sup_{\varphi \leq f} \int_E \varphi(x)dx$

B. State Fatou's lemma and use it to prove the Monotone Convergence theorem. Give an example to show that the that Monotone Convergence theorem will not hold for a decreasing sequence of non negative measurable functions.

19) A. State and prove the Lebesgue convergence theorem for measurable functions

B. Let f be an increasing real valued function on the interval $[a, b]$. Then show that f is differentiable almost everywhere, the derivative f' is measurable and

$$\int_a^b f'(x)dx \leq f(b) - F(a)$$

20) State and prove the Radon - Nikodym theorem.

21) Let μ^* be an outer measure defined on the power set of a nonempty set X. Show that the class $\mathfrak{B}$ of all μ^* - measurable subsets is a σ - algebra. If $\overline{\mu}$ is the restriction of μ^* to $\mathfrak{B}$, show that $\overline{\mu}$ is a complete measure on $\mathfrak{B}$

22) If $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ – finite measure spaces and if $V \in \mathcal{S} \times \mathcal{T}$, $\varphi(x) = \nu(V_x)$ and $\psi(y) = \mu(V^y)$ for each $x \epsilon X$ and $y \epsilon Y$, show that φ is $\mathcal{S}$ measurable ψ is $\mathcal{T}$ measurable and $\int_X \varphi \, d\mu = \int_X \psi \, d\nu$