MSc. Mathematics Degree (MGU-CSS-PG) Examination

(Model Question)

Ist Semester

PC 1-MT01C01 - LINEAR ALGEBRA

Time 3 hrs. Maximum Weight. 30

PART-A

Answer any 5 . Each question has 1 weight

1.) Define vector space.Let V be the set of pairs (x,y) of real numbers and let F be the field of real numbers.Define $(x,y) + (x_1,y_1) = (0, y + y_1)$

$$c(x,y) = (cx,cy).$$

Is V with these operations a vector space?

2)Let V be the vector space of all 2x2 matrices over the field F.Let W_1 be the set of matrices of the form $\begin{pmatrix} x & -x \\ Y & z \end{pmatrix}$ and let W_2 be the set of matrices of the form $\begin{pmatrix} a & b \\ -a & c \end{pmatrix}$

(i)Prove that W_1 and W_2 are subspaces of V

(ii)Find the dimension of $W_1 \cap W_2$

3)Let T be the linear operator on R^2 defined by $T(x_1,x_2) = (-x_2 , x_1)$.What is the matrix of T in the standard ordered basis for R^2? Further prove that for every real number 'c' the operator T- cI is invertible.

4) Define the dual space V* of a vector space over the field F.If V has finite dimension n, prove that dimV* = n by finding a basis for V*.

5) Let D be an n-linear function on n x n matrices over K.Suppose D has the property that D(A) = 0 whenever two adjacent rows of A are equal.Then prove that D is alternating.

6) Let K be a commutative ring with identity and let A and B be n x n matrices over K. Then det(AB) = (detA) (detB)

7) If T^2 = T,show that T is diagonalizable

8)If T is a linear operator on a finite dimensional vector space V, (a)Define (i)Characteristic Polynomial for T. (ii)Minimal polynomial of T

(b) Do similar matrices have the same minimal polynomial?Give reason.

PART-B

Answer any 5. Each question has 2 weights

9)Show that the vectors α_1 =(1,0,-1) , α_2 = (1,2,1) and α_3 = (0,-3,2) form a basis for R^3. Express each of the standard basis vectors of R^3 as a linear combination of α_1 ,α_2 and α_3.

10) Let V be a finite dimensional vector space over the field F and let $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ be an ordered basis for V.Let W be a vector space over the same field F and let $\beta_1,\beta_2,..\beta_n$ be any vectors in W.Then prove that there is precisely one linear transformation T from V into W such that $T\alpha_j = \beta_j$, j =1,2,…n

11) Let V and W be finite dimensional vector spaces over the field F such that dimV=dimW.If is a linear transformation from V into W, then prove that the following are equivalent:

(i)T is invertible

(ii)T is non-singular

(iii)T is onto

12) (a)Define with examples: (i)Transpose of alinear transformation (ii)double dual

(b)Let V be a finite dimensional vector space over F.Show that each basis of V* is the dual of some basis for V.

13) Let K be a commutative ring with identity.Show that the determinant function on 2x2 matrices A over K is alternating and 2-linear as function of columns of A.

14) (a)Define invariant subspaces

(b)Let T be a linear operator on V.Let U be any linear operator on V which commutes with T.Let W be the range of U and N be the null space of U,Then prove that W and N are invariant under T.

15) Let T be a linear operator on an n-dimensional vector space V.Prove that the characteristic and minimal polynomial for T have the same roots except for multiplicities.

16) Let T be the linear operator on R^2,the matrix of which in the standard ordered basis is $\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$ Find all subspaces of R^2 that invariant under T

PART-C

Answer any 3. Each question has 3 weights

17) Let W be the subspace of C^3 spanned by α_1 = (1,0,i) and α_2 = (1+i ,1, -1)

(i)Show that α_1 and α_2 form a basis for W.

(ii)Show that the vectors β_1 = (1,1,0) and β_2 = (1, i, 1+i) are in W and form another basis for W.

(iii)What are the co-ordinates of α_1 and α_2 in the ordered basis $\{\beta_1 , \beta_2\}$ for W.

18) (a)Find the subspace annihilated by the following functionals on R^4

$f(x_1,x_2,x_3,x_4) = x_1 +2x_2 + 2x_3 + x_4$

$g(x_1,x_2,x_3,x_4) = 2x_2 +x_4$

$h(x_1,x_2,x_3,x_4) = -2x_1 - 4x_3 + 3x_4$

(b)Let T: V$\rightarrow$W be linear where V and W are vector spaces over F.Show that

(i)The range(T^t) is the annihilator of the null space of T.

(ii)Rank(T^t) = Rank(T)

19) (a)Prove that if f is a non-zero linear functional on the vector space V,then the null space of f is a hyperspace in V and conversely every hyper space in V is the null space of a non-zero linear functional on V.

(b)Let g, f_1, f_2, ….,f_r be linear functionals on a vector space V with respective null spaces N_1,N_2,… ,N_r. Then prove that g is a linear combination of f_1, f_2, ….,f_r if and only if N contains the intersection $N_1 \cap N_2 \ldots . \cap N_r$.

20) If D is any alternating n-linear function on K^{nxn} then prove that for each nxn matrix A

D(A) = (detA) D(I) where I denotes the n x n identity matrix.

21) State and prove Cayley-Hamilton theorem for linear operators.

22) Let V be a finite dimensional vector space over the field F and let T be a linear operator on V.Prove that T is diagonalizable iff the minimal polynomial for T has the form $p = (x - c_1)(x - c_2)\ldots\ldots(x - c_k)$ where $c_1, c_2, \ldots c_k$ are distinct elements of F.

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